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THESIS

A COMPARISON OF SOLUTIONS OF A LINEAR HOMOGENEOUS SELF-ADJOINT DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS BY THE NEWTON, STODOLA AND RAYLEIGH-RITZ METHODS

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ABSTRACT

Three techniques for finding the eigenvalues and eigenfunctions are investigated. A typical problem involves a linear homogeneous differential equation with variable coefficients of the form

$$P(x)\, y''(x) + P'(x)\, y'(x) + \omega^2 M(x) = 0 \,. \qquad (1)$$

The functions $P(x)$ and $M(x)$ are functions which are positive, or have at most isolated zeroes on the fundamental interval $(0,L)$; ω is a parameter. Appropriate end conditions are specified so that the problem is self-adjoint.

The three methods are: Newton's method, Stodola's method, and the Rayleigh-Ritz method. The methods are derived and a computer solution by each method is included in the paper. A second problem involving Bessel's equation of order zero is solved using each method and a comparison of the eigenvalues and eigenfunctions is made with tabulated values.

The results indicate that Newton's method is to be preferred usually.

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I. INTRODUCTION

Consider the linear homogeneous differential equation with variable coefficients of the form

$$P(x)\, y''(x) + P'(x)\, y'(x) + \omega^2\, M(x)\, y(x) = 0, \qquad x \epsilon (0,L), \tag{1}$$

where $P(x)$ is of class C^1, $M(x)$ is continuous, and both functions are positive, or have at most isolated zeroes on the fundamental interval $(0,L)$. Also given are end conditions of the form

$$\begin{aligned} a_1\; y(x_1) + b_1\, y'(x_1) &= 0 \\ a_2\; y(x_2) + b_2\, y'(x_2) &= 0 \end{aligned} \tag{2}$$

The function y and the parameter ω are to be determined.

An equation of the form (1) may arise as a result of solving a partial differential equation of the form

$$M(x)\, \frac{\partial^2\, U(x,t)}{\partial\, t^2} = \frac{\partial}{\partial x}\left(P(x)\, \frac{\partial U\; (x,t)}{\partial x}\right) \tag{3}$$

by separation of variables with boundary conditions corresponding to (2). The solution of (3) is assumed to be of the form

$$U(x,t) = X(t)\; T(t) \tag{4}$$

and standard techniques for separation of variables [8] are employed. Equation (1) may also occur as the Euler equation [1] of an isopermetric problem of the form

$$\int_0^L P(x)\, y'(x)^2\, dx = \text{MIN} \tag{5}$$

subject to the constraint

$$\int_0^L M(x)\, y(x)^2\, dx = 1 \ . \tag{6}$$

This is equivalent to solving the problem

$$\int_0^L P(x)\, y'(x)^2 dx \Big/ \int_0^L M(x)\, y(x)^2 dx = \text{MIN}. \tag{7}$$

In this case, the Euler equation is

$$\frac{d}{dx}(F_{y'}) - F_y = \frac{d}{dx}[P(x)\, y'(x)] + \omega^2 M(x)\, y(x) = 0 \tag{8}$$

where

$$F = f^0 - \omega^2 f^1 \tag{9}$$

and

$$\begin{aligned} f^0 &= P(x)\, y'(x)^2 \\ f^1 &= M(x)\, y(x)^2. \end{aligned} \tag{10}$$

Calculus of variations treats the minimization or maximization of functionals such as integrals. This paper considers three techniques for obtaining the eigenvalues and eigenfunctions of Eq. (1). These techniques are Newton's method, Stodola's method, and the Rayleigh-Ritz method [3, 2, 5].

Newton's method may be applied to the solution of either the isopermetric problem or the equation resulting from separation of variables. The principal problem is to obtain a value of the solution to the differential equation which will satisfy the end conditions imposed by Eq. (2).

Consider y as a function $y(x,\omega)$ on those extremals [1] from (x_1, y_1). Let the derivative of y with respect to ω be

$$S(x,\omega) = \frac{\partial\ y\ (x_1\omega)}{\partial\ \omega} \quad . \tag{11}$$

A correctional equation

$$\omega_{New} = \omega_{Old} - \frac{y(x_2,\omega) - y(x_2)}{S(x_2,\omega)} \tag{12}$$

is derived in Chapter II and the sequence of steps to obtain the solution is discussed there.

Now consider Eq. (1) written in the self-adjoint form

$$- \frac{d}{dx}\left[P(x)\ y'(x)\right] = \omega^2\ M(x)\ y(x) \tag{13}$$

subject to the constraints (2). In Stodola's method, consider (13) to have the form

$$L\ y = \omega^2\ y \tag{14}$$

where the operator L is defined

$$L\ y(x) = \frac{1}{M(x)}\ \frac{d}{dx}\left(P(x)\ y'(x)\right) \tag{15}$$

For the eigenvalues ω_i^2 and associated eigenfunction Xi,

$$L\ Xi = \omega_i^2\ Xi\ . \tag{16}$$

Taking into account the end conditions, define an inverse operator, L^{-1}, then

$$L^{-1}\ Xi = \frac{1}{\omega_i^2}\ Xi \tag{17}$$

Since differential equation (13) and the boundary conditions (2) are self-adjoint, there is an infinite set of orthogonal eigenfunctions, Xi, and an arbitrary function, Y_0, satisfying the end conditions (2), can be expanded in a series of eigenfunctions

$$Y_0 = a_1 X_1 + a_2 X_2 + \dots \tag{18}$$

Repeated application of the inverse operator (17) to Y_0 emphasizes the coefficient of X_1, and decreases the relative size of the other coefficients. The resulting function Y is the corresponding eigenfunction. Further details of Stodola's method are discussed in Chapter III.

The Rayleigh-Ritz method is a procedure for obtaining approximate solutions of variational problems. The procedure consists of assuming that the desired extremal, y(x) can be approximated by a linear combination of n suitably chosen functions

$$y(x) \approx C_1 \Phi_1(x) + C_2 \Phi_2(x) + \dots + C_M \Phi_M(x) ; \tag{19}$$

The C_i are to be determined to effect the desired minimum. Usually the $\Phi\, i(x)$ are chosen to meet the boundary conditions for any choice of C_i. An approximation of y(x) may also be made by spline functions of the form

$$y(x) \approx C_1 + C_2 x + C_3 x^2 + C_4 \left[(x - \alpha_1)\right] + \dots + C_{K+3} \left[(x - \alpha_K)^2\right]$$

where the C_i are to be determined and the α_i are in (0,L). The function $\left[(x - \alpha_K)^2\right] = (x - \alpha_K)^2$ if $x > \alpha_K$ and is zero for $x < \alpha_K$.

The C's are not independent but must be chosen to satisfy the end conditions. In both the polynomial and spline function approximations, determination of the C_i is accomplished by solving an algebraic eigenvalue problem [4].

In the following chapters, Newton's, Stodola's and the Rayleigh-Ritz mentod are discussed in more detail and computer solutions of (1) by the three methods are presented.

II. NEWTON'S METHOD

The problem of minimizing the numerator of Eq. (1.7) with the denominator constrained to equal one is again being considered. The problem may be expressed in the form

$$I = \int_0^L P(x)\, y'(x)^2\, dx = \int_0^L f^0 dx = \text{MIN} \tag{1}$$

and

$$J = \int_0^L M(x)\, y(x)^2 dx = \int_0^L f^1\, dx = 1 \tag{2}$$

Assume the desired solution $y(x) = y^*(x)$ has been found. Then

$$\begin{aligned} a_1\, y(x_1) + b_1\, y'(x_1) &= 0 \\ a_2\, y(x_2) + b_2\, y'(x_2) &= 0 \end{aligned} \tag{3}$$

and

$$\int_0^L f^1(x,y,y')\, dx = 1 \quad , \quad y = y^* \quad , \quad y' = y^{*'} \tag{4}$$

A curve satisfying conditions (3) and (4) is called admissable.

A. DERIVATION OF NECESSARY CONDITION

The first necessary condition is derived for the case where $b_1 = 0 = b_2$. Consider replacing $y^*(x)$ by $y^*(x) + e\eta'(x)$, where $\eta(x)$ is an arbitrary function vanishing at 0 and L and having piecewise continuous first derivatives. It is necessary that

$$\frac{dI}{de} = \int_0^L \left(f^0_{y'} - \int_0^x f^0_y\, dx\right) \eta'(x)\, dx = 0 \tag{5}$$

or

$$\frac{dI}{de} = \int_0^L M_0(x)\ \eta'(x)\ dx = 0 \tag{6}$$

for all η such that

$$\frac{dJ}{de} = \int_0^L \left(f^1_{y'} - \int_0^x f^1_y\ dx\right) \eta'(x)\ dx = 0 \tag{7}$$

or

$$\frac{dJ}{de} = \int_0^L M_1(x)\ \eta'(x)\ dx = 0 \tag{8}$$

In Eq. (6) and (8), $M_i(x)$ is defined as

$$M_i(x) = \left(f^i_{y'} - \int_0^x f^i_y\ dx\right)_{y\,=\,y^*\ ,\ \ y'\,=\,y^{*\prime}} . \tag{9}$$

Let

$$g_i(x) = M_i(x) - M_{i_{av}} = M_i(x) - \int_0^L M_i(x)\ dx/L . \tag{10}$$

The above necessary condition may be reexpressed. It is necessary that

$$\frac{dI}{de} = \int_0^L g_0(x)\ \eta'(x)\ dx = 0 \tag{11}$$

for all η such that

$$\frac{dJ}{de} = \int_0^L g_1(x)\ \eta'(x)\ dx = 0 \tag{12}$$

since

$$\int_0^L M_{i_{av}}\ \eta'(x)\ dx = M_{i_{av}} \int_0^L \eta'(x)\ dx = M_{i_{av}}\ \eta(x) \Big|_0^L = 0 \tag{13}$$

Now choose

$$\eta'(x) = g_0(x) + c\ g_1(x) \tag{14}$$

and choose c so that $\eta'(x)$ is admissable:

$$\frac{dJ}{de} = \int_0^L g_1(g_0 + cg_1)\ dx = 0\ . \tag{15}$$

If $g_1(x) \neq 0$, Eq. (16) can be solved for c giving

$$c = -\int_0^L g_0(x)\ g_1(x)\ dx \Big/ \int_0^L g_0(x)^2\ dx \tag{16}$$

Now consider

$$\frac{dI}{de} = \int_0^L g_0(x)\ \Big(g_0(x) + c\ g_1(x)\Big)\ dx \tag{17}$$

then

$$\begin{aligned}\frac{dI}{de} + c\,\frac{dJ}{de} &= \int_0^L \Big(g_0(x) + c\ g_1(x)\Big)\Big(g_0(x) + c\ g_1(x)\Big)\ dx \\ &= \int_0^L \Big(g_0(x) + c\ g_1(x)\Big)^2\ dx \geq 0\ .\end{aligned} \tag{18}$$

Then

$$\frac{dI}{de} > 0 \tag{19}$$

unless

$$g_0(x) + c\ g_1(x) = 0\ . \tag{20}$$

Hence this is a necessary condition, known as the First Necessary Condition, or the integrated form of the Euler equation. Writing Eq. (20) as

$$f^0_{y'} + c\ f^1_{y'} - \int_0^x (f^0_y + c\ f^1_y)\ dx = \Big(M_0(x) + c\ M_1(x)\Big)_{av} \tag{21}$$

it is seen that f^0 and f^1 enter only on the combination

$$F = f^0 + c\ f^1\ . \tag{22}$$

The integral form of Euler's equation [1]

$$F_{y'} - \int_0^x F_y \, dx = M_{av} = \text{CONSTANT} \tag{23}$$

is obtained from (20) by setting

$$M_{av} = M_{0_{av}} + c \, M_{1_{av}} \; . \tag{24}$$

Now let

$$c = - \omega^2 \tag{25}$$

then the Euler Equation

$$F_{y'y'} \, y''(x) + F_{y'y} \, y'(x) + F_{y\,x} \; y(x) - F_y = 0 \tag{26}$$

becomes

$$P(x) \; y''(x) + P'(x) \; y'(x) = - \omega^2 \; M(x) \; y(x) \; . \tag{27}$$

It may be noted that this differential equation is self-adjoint.

B. COMPUTATIONAL ROUTINE

It will generally be necessary to generate the curve by forward numerical integration. There are two parameters

$$c_1 = c = - \omega^2 \tag{28}$$

and

$$c_2 = y'(0) \tag{29}$$

which are needed to determine a curve which is an approximation to the solution; an estimate of $y_2'(0)$ is needed in order to start integration.

In order to apply Newton's method it is necessary to evaluate derivatives with respect to these parameters. Let

$$u = \frac{\partial\ y(x)}{\partial\ c_1} \tag{30}$$

and

$$v = \frac{\partial\ y(x)}{\partial\ c_2}\ . \tag{31}$$

Then differentiating Eq. (27) with respect to c_1 gives

$$P(x)\ u'' + P'(x)\ u' - c_1\ M(x)\ u = M(x)\ y(x) \tag{32}$$

Denote the left side of (32) as Lu, defining the second-order linear differential operator L. Then the equations for u may be written

$$L\ u = M(x)\ y(x) \tag{33}$$

and

$$\begin{aligned} u(0) &= 0 \\ u'(0) &= 0 \end{aligned} \tag{34}$$

The corresponding equations for v are

$$P(x)\ v'' + P'(x)\ V' - c_1\ M(x)\ v = 0 \tag{35}$$

and

$$\begin{aligned} v(0) &= 0 \\ v'(0) &= 1 \end{aligned} \tag{36}$$

or

$$L\,v = 0$$

$$v(0) = 0 \qquad (37)$$

$$v'(0) = 1\ .$$

The correctional equation for J is obtained from the approximate relations

$$\Delta J = \frac{\partial J}{\partial c_1}\,\Delta c_1 + \frac{\partial J}{\partial c_2}\,\Delta c_2 \qquad (38)$$

$$= A_1 - J\ ;$$

by J is meant the computed value. The correctional equation for y is obtained from the linear approximation

$$\Delta\, y(L) = u(L)\,\Delta c_1 + v(L)\,\Delta c_2 \qquad (39)$$

$$= -\,y(L)$$

where $y(L)$ is the computed value.

However this problem has a peculiarity that makes it somewhat simpler. Because both integrals are quadratic and homogeneous in the pair $y(x)$ and $y'(x)$, the principal problem is to find c_1 so that $y(L) = 0$. The constant c_2 is then obtained by substituting the solution y into J:

$$c_2 = 1/\sqrt{J} \qquad (40)$$

Consider then y as a function of a single parameter

$$y = y(x,\ c_1) \qquad (41)$$

The correctional equation reduces to

$$\Delta y = u(L) \Delta c_1 \tag{42}$$

$$= y(L)/u(L)$$

or

$$\Delta \omega^2 = -y(L)/u(L) \tag{43}$$

In the computational routine, an initial estimate of ω^2 is made. Then $y(x)$ and u are obtained by integrating Eq. (33). The correction for ω^2 is then obtained from Eq. (43). If a good guess for ω^2 is made, this method converges rapidly. If a normalized value of $y(x)$ is desired then a new solution may be obtained from

$$y(x)_{new} = y(x)/\left(\int_0^L M(x)\ y(x)^2\ dx\right)^{\frac{1}{2}} \tag{44}$$

To tell if the lowest eigenfunction has been obtained, a check is made to see if the eigenfunction obtained has a zero in (0,L). If it has no zero in (0,L), the lowest eigenfunction has been obtained.

III. STODOLA'S METHOD

Consider Eq. (1.1)

$$\frac{d}{dx}\Big(P(x)\, y'(x)\Big) = -\omega^2\, M(x)\, y(x), \tag{1}$$

subject to the constraints

$$\begin{aligned} a_1\, y(0) + b_1\, y'(0) &= 0 \\ a_2\, y(L) + b_2\, y'(L) &= 0 \end{aligned} \tag{2}$$

Stodola's method of finding the lowest eigenvalue and eigenfunction will now be developed. The system of Eqs. (1) and (2) is self-adjoint. In this case there are an infinite number of eigenvalues $\lambda i = \omega_i^{\,2}$, $i = 1, 2, \ldots$ and associated eigenfunctions X_i.

Consider Eq. (1) as having the form

$$L\, y(x) = \omega^2\, y(x) \tag{3}$$

in which the linear differential operator L is defined by

$$L\, y(x) = \frac{-1}{M(x)}\, \frac{d}{dx}\Big(P(x)\, y'(x)\Big) \tag{4}$$

and consider any particular eigenfunction Xi. Then

$$\begin{aligned} L\ \ Xi &= \omega_i^2\ \ Xi \\ L^2\ Xi &= \omega_i^4\ \ Xi \\ &\ \vdots \\ L^m\ Xi &= \omega_i^{2m}\ Xi \end{aligned} \tag{5}$$

The inverse operator, L^{-1}, is now defined, taking into account the end conditions (2). Consider

$$\frac{d}{dx}\left(P(x)\ Xi'\right) = -\ \omega_i^2\ M(x)\ Xi \tag{6}$$

and integrate (6) to obtain

$$P(x)\ X_i' = -\ \omega_i^2 \left(\int_0^x M(x)\ Xi\ dx - c\right); \tag{7}$$

the constant c is to be determined later. The eigenfunction, X_i, may be obtained by dividing Eq. (7) by P(x) and integrating again

$$X_i = -\ \omega_i^2 \left[\int_0^x \frac{\int_0^x M(x)\ X_i\ dx}{P(x)}\ dx - \int_0^x \frac{c}{P(x)}\ dx - d\right]. \tag{8}$$

The constants of integration, c and d, may be obtained from the constraints in Eq. (2), and Eqs. (7) and (8). Solving these equations gives

$$c = \frac{\left[a_2 \int_0^L \dfrac{\int_0^L M(x)\ X_i\ dx}{P(x)}\ dx + \dfrac{b_2 \int_0^L M(x)\ X_i\ dx}{P(L)}\right]}{\left[a_2 \int_0^L \dfrac{dx}{P(x)} + \dfrac{b_2}{P(L)} - \dfrac{a_2\ b_1}{a_1\ P(0)}\right]} \tag{9}$$

and

$$d = -\ b_1\ d/a_1\ P(0) \tag{10}$$

if $a_1 \neq 0$. If $a_1 = 0$ or $P(0) = 0$ then $c = 0$ and a solution for d is obtained.

The set of operations on the right side in Eqs. (7) and (8) define the inverse operator, L^{-1}, with c and d given by Eqs. (9) and (10). These operations give

$$X_i = \omega_i^2 L^{-1} X_i \tag{11}$$

or

$$L^{-1} X_i = X_i/\omega_i^2 \tag{12}$$

Stodola's method makes use of the fact that an arbitrary piecewise-continuous function which satisfies the end conditions (2) can be expanded in a series on the eigenfunctions. The first estimate Yo of the first eigenfunction may be chosen as follows. Let Yo be the first estimate of X_i: choose it so that it has no zeroes inside the interval (0,L). Then Yo may be considered to be the following form

$$Yo = a_1 X_i + a_2 X_2 + \dots \tag{13}$$

where a_1 is not equal to zero. For convenience Yo is normalized; the L_∞ norm of Yo, $||Yo||$, is defined as

$$\max_{0 < X < L} |Yo(x)| = ||Yo|| \, . \tag{14}$$

and $||Yo||$ is set equal to one. Now consider

$$L^{-1}Yo = \frac{a_1}{\omega_1^2} X_1 + \frac{a_2}{\omega_2^2} X_2 + \dots \tag{15}$$

and

$$L^{-m}Yo = \frac{a_1}{\omega_1^{2m}} X_1 + \frac{a_2}{\omega_2^{2m}} X_2 + \dots \tag{16}$$

$$= \frac{1}{\omega_1^{2m}} \left(a_1 X_1 + a_2 \left(\frac{\omega_1}{\omega_2}\right)^{2m} X_2 + a_3 \left(\frac{\omega_1}{\omega_3}\right)^{2m} X_3 + \dots \right)$$

From Eq. (16) it can be seen that the relative size of the components X_2, X_3, ... is decreasing by a factor $\left(\frac{\omega_1}{\omega_2}\right)^2$, $\left(\frac{\omega_1}{\omega_3}\right)^2$, ... respectively. After a few applications of the operator, L^{-1}, to Eq. (13), the leading term will dominate. It is convenient to normalize after each iteration; let

$$Z_m = L^{-1} Y_{m-1} \tag{17}$$

where Ym is the m'th approximation to X_1 and

$$Ym = Z_m / ||Z_m|| . \tag{18}$$

When this is compared with Eq. (12), where X_i on the left hand side corresponds to Y_{m-1} and X_i on the right hand side corresponds to Y_m, it is seen that ω_1^2 is approximately

$$\omega_1^2 = 1/||Z_m|| . \tag{19}$$

The iteration is stopped when

$$||Y_m(x) - Y_{m-1}(x)|| < \epsilon , \tag{20}$$

where ϵ is some specified small number. The resulting function Y_m is an approximation to X_1.

In the computational routine an initial approximation, Y_o to X_1 is made which satisfies the constraints (2) and Y_o is normalized. The next approximation to X_1 is obtained by forward integration of Eqs. (7) and (8). The contrants of integration, c and d are determined by Eqs. (9) and (10). The value of ω_1^2 is approximately $1/||Z_m||$. The new approximation, Z_m, is normalized, Eq. (18), th procedure is

repeated. When the norm of the difference between successive iterates, Y_{m-1}, Y_m, is less than some prescribed value the routine is stopped. The value of ω^2 from Eq. (19) is taken as the lowest eigenvalue and Y_m is the corresponding eigenfunction. The numerical solution of Bessel's equation of order zero gave a good approximation to the eigenvalue but a smaller value of the eigenfunction.

IV. RAYLEIGH-RITZ METHOD

The Rayleigh-Ritz method has long been used to obtain an approximate solution to eigenvalue and eigenfunction problems. This is still a useful method if a high speed computer is not available.

Assume that m functions $\Phi_i(x)$ are given which satisfy the end conditions. These are often chosen as polynomials or trigonometric functions. The solution then is approximated by a linear combination of these

$$y = \sum_{i=1}^{m} c_i \Phi_i(x) \, , \tag{1}$$

This is substituted into

$$\omega^2 = \int_0^L P(x)\, y'(x)^2 \, dx \Big/ \int_0^L M(x)\, y(x)^2 \, dx = \mathrm{MIN} \tag{2}$$

and the c's are chosen to minimize (2). This is equivalent to the minimization of a quadratic form

$$\sum_{i=1}^{m} \sum_{i=1}^{m} a_{ij} \, c_i \, c_j \, ,$$

where

$$a_{ij} = \int_0^L P(x)\, \Phi_i'(x)\, \Phi_j'(x)\, dx \, , \tag{3}$$

subject to a constraint that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} \, c_i \, c_j = 1 \, , \tag{4}$$

where

$$b_{ij} = \int_0^L M(x)\ \Phi_i(x)\ \Phi_j(x)\ dx\ . \tag{5}$$

Equation (2) may then be written

$$\omega^2 = \frac{\overline{C}^T\ \overline{A}\ \overline{C}}{\overline{C}^T\ \overline{B}\ \overline{C}} = \text{MIN} \tag{6}$$

where

$$\overline{A} = a_{ij}\ ,\quad i = 1,2,\ldots,m\ ,\quad j = 1,2,\ldots,m, \tag{7}$$

$$\overline{B} = b_{ij}\ ,\quad i = 1,2,\ldots,m\ ,\quad j = 1,2,\ldots,m, \tag{8}$$

and

$$\overline{C} = (c_1,\ c_2,\ \ldots\ c_m)^T \tag{9}$$

The problem may be reduced as follows. Let the eigenvalues of B be λ_i^2 , $i = 1,2,\ldots m$ and the associated normalized eigenvectors u_i . Then

$$\overline{T}^T\ \overline{B}\ \overline{T} = \overline{I} \tag{10}$$

where

$$T = (u_1,\ u_2,\ \ldots,\ u_m)\ \text{DIAG}\ (1/\lambda_1,\ 1/\lambda_2,\ \ldots,\ 1/\lambda_m) \tag{11}$$

and the transformation

$$\overline{C} = \overline{T}\ \overline{D}, \tag{12}$$

where

$$\overline{D} = (d_1,\ d_2,\ \ldots,\ dm)^T \tag{13}$$

gives

$$\begin{aligned} \bar{D}^T \bar{D} &= \bar{D}^T \bar{T}^T \bar{B} \bar{T} \bar{D} \\ &= \bar{C}^T \bar{B} \bar{C} \\ &= 1 \end{aligned} \tag{14}$$

The desired minimum value of

$$\bar{C}^T \bar{A} \bar{C} = \bar{D}^T \bar{T}^T \bar{A} \bar{T} \bar{D} = \text{MIN} \tag{15}$$

subject to (14) is the minimum eigenvalue of

$$\bar{E} = \bar{T}^T \bar{A} \bar{T} \tag{16}$$

and the corresponding vector $\bar{D}$ is the associated normalized eigen-vector of $\bar{E}$. Since the elements of $\bar{T}$ and $\bar{D}$ are known, $\bar{C}$ may be evaluated from (4.12).

The minimization technique using the spline function approximation

$$y(x) \approx C_1 + C_2 X + C_3 X^2 + C_4 [(x - \alpha_1)^2] + \ldots + C_{K+3} [(x - \alpha_K)^2] \tag{17}$$

is similar to the polynomial approximation method. In this case however, the functions $\Phi_i(x)$ do not satisfy the end conditions. Hence the constants above are not independent but one of them is dependent on the others for the end conditions to be satisfied. It is necessary to have an initial subprogram to eliminate this constant. Then the problem is reduced to the same form as before

$$\sum_{i=1}^{m-1} \sum_{j=1}^{m-1} a_{ij} c_i c_j = \text{MIN} \tag{18}$$

and

$$\sum_{i=1}^{m-1} \sum_{j=1}^{m-1} b_{ij} c_i c_j = 1 \tag{19}$$

and solved as before.

V. APPLICATION OF RAYLEIGH-RITZ, STODOLA AND NEWTON METHODS

In Chapters II, III and IV, three methods were developed for finding the lowest eigenvalue and eigenfunction for Eq. (1.1). These methods were used to obtain numerical solutions of the following two problems: Bessel's equation of order zero

$$X\, y''(x) + y'(x) + \omega^2\, x\, y(x) = 0 \,, \quad x \epsilon (0, 2.405), \tag{1}$$

subject to the constraints

$$y(0) = 1 \,, \quad y(2.405) = 0 \,, \quad y'(0) = 0 \,; \tag{2}$$

and an equation of the form

$$(1+x)\, y''(x) + y'(x) + \omega^2\, y(x) = 0 \,, \quad x \epsilon (0,1) \,, \tag{3}$$

subject to the constraints

$$y(0) = y(1) = 0 \,. \tag{4}$$

In this chapter the computation and numerical results are discussed.

In the Rayleigh-Ritz polynomial approximation, an approximation for the extremal, $y(x)$, satisfying Eq. (3) was made by choosing suitable functions $\Phi_1(x)$ and $\Phi_2(x)$ to satisfy the constraints in Eq. (4). The approximation for $y(x)$ is

$$\begin{aligned} y(x) &= c_1\, \Phi_1(x) + c_2\, \Phi_2(x) \qquad (5) \\ &= c_1\, x\, (1-x) + c_2\, x^2\, (1-x) \end{aligned}$$

The values of c_1, c_2, and the lowest eigenvalues were obtained by the techniques described in Chapter II. Computer Output 1. shows the values of c_1 and c_2, the lowest eigenvalue and the values of y(x) at tenths of an interval. The numerical solution of Eq. (3) using the spline function approximation

$$y(x) \approx c_1 \, x + c_2 \, x^2 + c_3 \left[(x-1/3)^2\right] + c_4 \left[(x-2/3)^2\right] \tag{6}$$

is shown in Computer Output 2. and Computer Program 2. This solution followed a format similar to the polynomial approximation solution. However, a transformation was required to eliminate the dependent coefficient. In this case the coefficient c_4 was taken as dependent on c_1, c_2 and c_3. The required matrix products, transpositions and eigenvalue solutions were evaluated by using subroutine GMPRD, GMTRA, and EIGEN respectively, [4].

A numerical solution of Eq. (3) by Stodola's method using as a first estimate

$$y(x) = 4x\,(1-x) \tag{7}$$

to satisfy the constraints in Eq. (4) yielded the values in Computer Output 3. The solution of Eq. (3) by Newton's method with an initial estimate of

$$\omega^2 = 2.0 \tag{8}$$

is shown in Computer Output 4. In these two methods the fundamental interval (0,1) was sub-divided into one hundred equal sub-intervals to perform the second order Runga-Kutta integration routine. If the error between the computed value and the boundary value at X equal

one was less than one ten thousandth, the program was terminated. The programs for Stodola's and Newton's method are shown in Computer Program 3. and Computer Program 4.

The lowest eigenvalues obtained by the above methods are reasonably close in value. The maximum difference in values was between Newton's method

$$\omega = 3.78634 \tag{9}$$

and Rayleigh-Ritz polynomial approximation,

$$\omega = 3.79813 \ . \tag{10}$$

Newton's method converged in three iterations and Stodola's method required seven iterations.

The Rayleigh-Ritz, Stodola and Newton methods were used to find a numerical solution to Bessel's equation of order zero, Eq. (1). A tabulated solution, [7], of Eq. (1) gives the smallest eigenvalue as one. The result obtained by the Rayleigh-Ritz method was

$$\omega = 1.01435 \tag{11}$$

when the function was approximated by quadratic splines

$$y(x) = c_1 + c_2 X^2 + c_3 \left[(x - 0.8)^2\right] + c_4 \left[(x - 1.6)^2\right] . \tag{12}$$

The smallest eigenvalue of Eq. (1) obtained by using Stodola's method with an initial estimate of

$$y(x) = \left((1-x)/2.405\right)^2 \tag{13}$$

is

$$\omega = 1.00000 \tag{14}$$

In Newton's method an initial estimate of

$$\omega = 1.4 \tag{15}$$

yielded a value

$$\omega = 1.0000 \tag{16}$$

for the lowest eigenvalue.

A comparison of the eigenfunction, $y(x)$, obtained by the three methods was made with tabulated values for $y(x)$ of Eq. (1). The values obtained by Newton's method at tenths of an interval were within five ten-thousandths, by Rayleigh-Ritz were within five thousandths and by Stodola's method were within two one-hundredths. The best approximation of the lowest eigenfunction was obtained by Newton's method.

The solution of ω^2 obtained by the Rayleigh-Ritz method is greater than the theoretical value as given in Reference 5. The numerical solution of Bessel's equation of order zero supported this conclusion. The value obtained for ω^2 was approximately one-thousandth larger than the theoretical value.

The results of the three methods are shown in Computer Output 5., 6., and 7. and the programs are shown in Computer Programs 5., 6., and 7.

VI. CONCLUSION

The three methods each yield satisfactory values for the eigenvalues and the eigenvectors. Generally, Newton's method seems to be most satisfactory. It is for the most part straightforward to program. The only significant errors that are not apparent are there due to the integration routine. It is very flexible, being applicable to many different types of problems. It converges rapidly if a good initial estimate is made, and computing times seem to be generally small. Proof of the exact convergence of Newton's method may be found in Reference 3.

Stodola's method also is quite satisfactory. There seems to be no question of convergence. The initial guess can be quite poor and it will still converge. More iterations than for Newton's method seems to be required to get the same accuracy, and there is no simple way to get an estimate of the error.

The Rayleigh-Ritz method seems to have little to recommend it if a good computer is available. It is more difficult to program. It has one advantage that no iteration is required. Use of spline functions increases the tedium of programming so that they are not worthwhile.

The methods of Stodola and Rayleigh-Ritz were used before the advent of large scale computers, though application was rather limited compared to problems that can be treated now.

In case larger eigenvalues and associated eigenfunctions are needed, Newton's method yields these directly if the initial estimate of ω is changed. The number of zeros of the eigenfunction obtained show which eigenvalue and eigenfunction has been obtained. The eigenfunctions are automatically orthogonal with a weight factor $M(x)$ since the system is self-adjoint.

If Stodola's method is used to obtain higher eigenvalues and eigenfunctions it is necessary to adjoin a further condition that each new eigenfunction be orthogonal to all of the preceeding. This increases the tedium of programming and decreases the accuracy, since preceding values are only approximate.

In the Rayleigh-Ritz method, an estimate of other eigenvalues and eigenfunctions is obtained from the other eigenvectors of the matrix $\bar{E}$ of chapter IV. The eigenvalues are too high unless the eigenfunction can be expressed as a finite sum of the functions $\Phi_i(x)$.

COMPUTER OUTPUT 1. (Polynomial)

THE SMALLEST EIGENVALUE 3.798129

THE VALUES OF C1 AND C2

C1= 7.014201

C2= -3.138530

THE VALUES OF Y(X) AT ONE-TENTH INTERVAL

X	Y
0.0	0.0
0.100000	0.603039
0.200000	1.021852
0.300000	1.275272
0.400000	1.382130
0.500000	1.361256
0.599999	1.231482
0.699999	1.011638
0.799999	0.720556
0.899999	0.377068
0.999999	0.00004

COMPUTER OUTPUT 2. (Spline)

```
THE SMALLEST EIGENVALUE        3.790108

VALUES OF C1,C2,C3,AND C4

C1=          6.358921

C2=         -7.197122

C3=          0.565650

C4=          5.281205

VALUES OF Y(X) AT ONE-TENTH INTERVAL

    X                  Y
0.0                  0.0

0.100000             0.563920

0.200000             0.983899

0.300000             1.259933

0.400000             1.394543

0.500000             1.395892

0.599999             1.264613

0.699999             1.006572

0.799999             0.698055

0.899999             0.362533

0.999999             0.000003
```

COMPUTER OUTPUT 3. (Stodola)

```
 NI= 7.OMEGA =     3.787142
THE MAXIMUM DIFFERENCE BETWEEN YB AND YBN IS     0.000047
PE,KE,AND ROOT (PE/KE) ARE    7.172087    0.498484    3.793123
```

X	YBN
0.0	0.0
0.100000	0.387621
0.200000	0.694288
0.300000	0.897866
0.400000	0.992039
0.500000	0.982383
0.600000	0.883094
0.700000	0.713473
0.800000	0.495322
0.900000	0.250609
1.000000	-0.000014

COMPUTER OUTPUT 4. (Newton)

THIS IS PATH NO. 3. OMEGA= 3.78634

X	Y
0.0	0.0
0.100000	0.093143
0.200000	0.166838
0.300000	0.215768
0.400000	0.238397
0.500000	0.236090
0.600000	0.212242
0.700000	0.171498
0.800000	0.119087
0.900000	0.062299
1.000000	0.000081

COMPUTER OUTPUT 5. (Spline)

THE SMALLEST EIGENVALUE 1.000103

VALUES OF C1,C2,C3,AND C4

C1= -1.132401

C2= 0.272184

C3= -0.125127

C4= -0.184540

VALUES OF Y(X) AT ONE-TENTH INTERVAL

X	Y
0.0	-1.132401
0.240500	-1.116657
0.481000	-1.069427
0.721500	-0.990712
0.962000	-0.883794
1.202499	-0.759094
1.443000	-0.617383
1.683499	-0.459947
1.924000	-0.302298
2.164499	-0.148986
2.405000	-0.000009

COMPUTER OUTPUT 6. (Stcdola)

```
 NI= 7. CMEGA=     1.000000
THE MAXIMUM DIFFERENCE BETWEEN YB AND YBN IS     0.000021
PE,KE,AND ROOT (PE/KE) ARE     0.755341     0.761183     0.996155
```

X	YBN
0.0	1.000000
0.240500	0.974024
0.481000	0.931927
0.721500	0.863787
0.962000	0.772530
1.202499	0.662059
1.443000	0.537061
1.683499	0.402789
1.924000	0.264808
2.164499	0.128741
2.405000	0.0

COMPUTER OUTPUT 7. (Newton)

THIS IS PATH NO. 4. OMEGA = 0.99998

X	Y
0.0	1.000000
0.240500	0.985594
0.481000	0.942999
0.721500	0.874051
0.962000	0.781713
1.202499	0.669932
1.443000	0.543451
1.683499	0.407585
1.924000	0.267966
2.164499	0.130282
2.405000	0.000010

COMPUTER PROGRAM 1.

RAYLEIGH-RITZ METHOD APPLIED TO EQN (1.1)

(1+X)*Y''(X)+Y'(X)+(W**2)*Y(X)=0

```
      DIMENSION PH(2,101),PHP(2,101),A(2,2),B(2,2),P(101),
     1FA(101),FB(101),BL(2,2),AMB(2,2),C(2),AX(3),BX(3),
     2RLB(2),BXD(2),T(2,2),D(2),TT(2,2),ALPHA(2,2),AP(2,2),
     3S(2,2),RLBD(2),Y(101),Z(101)
      REAL LEV
      X=0.
      DX=0.01
      DO 1 I=1,101
C DEFINITION OF ELEMENTS OF MATRIX B
      P(I)=1.0+X
      PH(1,I)=X*(1.0-X)
      PH(2,I)=(X**2)*(1.0-X)
C DEFINITION OF ELEMENTS OF MATRIX A
      PHP(1,I)=1.0-2.0*X
      PHP(2,I)=2.0*X-3.0*(X**2)
      X=X+0.01
    1 CONTINUE
C COMPUTE A(I,K) AND B(I,K)
      DO 2 I=1,2
      DO 3 K=1,2
      FAINT =0
      FBINT =0
      FA(1)=P(1)*PHP(I,1)*PHP(K,1)
      FB(1)=PH(I,1)*PH(K,1)
      DO  4 J=2,101
      FA(J)=P(J)*PHP(I,J)*PHP(K,J)
      FB(J)=PH(I,J)*PH(K,J)
      FAINT=FAINT +(FA(J)+ FA(J-1))*DX/2
      FBINT=FBINT+ (FB(J)+FB(J-1))*DX/2
    4 CONTINUE
      A(I,K)=FAINT
      B(I,K)=FBINT
    3 CONTINUE
    2 CONTINUE
C INITIALIZE BX FOR THE INPUT TO EIGEN ROUTINE
      NC=1
      DO 201 J=1,2
      DO 202 I=1,2
      BX(NC)=B(I,J)
      NC=NC+1
      IF (I.GE.J) GO TO 201
  202 CONTINUE
  201 CONTINUE
      CALL EIGEN (BX,S,2,0)
C RLB(I) IS 1/SQRT OF EVAL OF B
      KK=1
      DO 8 II=1,2
      BXD(II)=BX(KK)
      KK=KK+II+1
    8 CONTINUE
      DO 84 II=1,2
      DO 9 JJ=1,2
      B (II,JJ)=0.
    9 CONTINUE
   84 CONTINUE
      DO 6 I=1,2
      RLBD(I)=SQRT(BXD(I))
      RLB(I)=1.0/RLBD(I)
      B (I,I)=RLB(I)
```

```
    6 CONTINUE
C MULT   S(I,J)*B (I,J)=T(I,J)
      CALL GMPRD (S,B ,T,2,2,2)
C TRANSPOSE T=TT
      CALL GMTPA (T,TT,2,2)
C MULT   A*T=AP
      CALL GMPRD ( A,T,AP,2,2,2)
C MULT   TT*AP=ALPHA
      CALL GMPRD (TT,AP,ALPHA,2,2,2)
C FIND E VAL AND E VEC OF ALPHA
      NC=1
      DO 210 J=1,2
      DO 211 I=1,2
       AX(NC)=ALPHA(I,J)
      NC=NC+1
      IF (I.GE.J) GO TO 210
  211 CONTINUE
  210 CONTINUE
      CALL EIGEN (AX    ,S,2,0)
      LEV=SQRT(AX(3))
      DO 7 I=1,2
      D(I)= S(I,2)
    7 CONTINUE
C MULT   T*D TO GET VALUES OF C
      CALL GMPRD (T,D,C,2,2,1)
C EVALUATE Y AND Z
      Z(1)=0
      DO 12 I=2,101
      Z(I)=Z(I-1)+0.01
   12 CONTINUE
      DO 13 I=1,101
      Y(I)=C(1)*PH(1,I)+C(2)*PH(2,I)
   13 CONTINUE
      WRITE (6.10 ) LEV,C(1),C(2),(Z(I),Y(I),I=1,101,10)
   10 FORMAT ('1'/////////,'0',15X,'THE SMALLEST EIGENVALUE',
     1F15.6//,'0',15X,'THE VALUES OF C1 AND C2'//,' ',15X,
     2'C1=',F15.6//,'0',15X,'C2=',F15.6//,'0',15X,
     3'THE VALUES OF Y(X) AT ONE-TENTH INTERVAL'//,
     4'0',18X,'X',15X,'Y'//('             ',2F15.6//))
      WRITE (6,998)
  998 FORMAT ('1',2X)
      STOP
      END
```

COMPUTER PROGRAM 2.

RAYLEIGH-RITZ METHOD APPLIED TO EQN (1.1)

(1+X)*Y''(X)+Y'(X)+(W**2)*Y(X)=0

```
      DIMENSION A(3,3),B(3,3),AX(6)   ,BX(6)   ,R(3,3),RLB(3),
     1T(3,3),ALPHA(3,3),AP(4,4),TT(3,3),S(3,3),D(3),C(3),
     2BXD(3),RLBD(3),AQ(4,4),BP(4,4),CM(4,3),CMT(3,4),BCM(
     34,3),ACM(4,3),P(101),PH(4,101),PHP(4,101),FA(101)
     4,FB(101),Y(101),Z(101)
      REAL LEV
      X=0.
      DO 1 I=1,101
C DEFINITION OF ELEMENTS OF MATRIX B
      P(I)=1.+X
      PH(1,I)=X
      PH(2,I)=X**2
      PH(3,I)=((X-1./3.+ABS(X-1./3.))/2.)**2
      PH(4,I)=((X-2./3.+ABS(X-2./3.))/2.)**2
C DEFINITION OF ELEMENTS OF MATRIX A
      PHP(1,I)=1.
      PHP(2,I)=2.*X
      PHP(3,I)=X-1./3.+ABS(X-1./3.)
      PHP(4,I)=X-2./3.+ABS(X-2./3.)
   92 X=X+0.01
    1 CONTINUE
      DX=0.01
C COMPUTE AP(I,K) AND BP(I,K)
      DO 2 I=1,4
       DO 3 K=1,4
      FAINT=0.
      FBINT=0.
      FA(1)=P(1)*PHP(I,1)*PHP(K,1)
      FB(1)=PH(I,1)*PH(K,1)
      DO 4 J=2,101
      FA(J)=P(J)*PHP(I,J)*PHP(K,J)
      FB(J)=PH(I,J)*PH(K,J)
      FAINT=FAINT+(FA(J)+FA(J-1))*DX/2
      FBINT=FBINT+(FB(J)+FB(J-1))*DX/2
    4 CONTINUE
      AP(I,K)=FAINT
      BP(I,K)=FBINT
    3 CONTINUE
    2 CONTINUE
C INITIALIZE CM
      DO 51 KA=1,4
      DO 52 KB=1,3
      CM(KA,KB)=0.0
   52 CONTINUE
   51 CONTINUE
C EVALUATION OF THE TRANSFORMATION MATRIX CM
      DO 53 KA=1,3
      CM(KA,KA)=1.0
   53 CONTINUE
      CM(4,1)=-9.
      CM(4,2)=-9
      CM(4,3)=-4
      CALL GMTRA(CM,CMT,4,3)
C MAKE TRANSFORMATION CM (TRANSPOSED)*B*CM
      CALL GMPRD(BP,CM,BCM,4,4,3)
      CALL GMPRD(CMT,BCM,B,3,4,3)
C MAKE TRANSFORMATION CM(TRANSPOSED)*A*CM
      CALL GMPRD(AP,CM,ACM,4,4,3)
      CALL GMPRD(CMT,ACM,A,3,4,3)
      NC=1
C INITIALIZE BX FOR THE INPUT TO EIGEN ROUTINE
      DO 201 J=1,3
```

COMPUTER PROGRAM 3.

STODOLA'S METHOD APPLIED TO EQN(1.1)

(1+X)*Y''(X)+Y'(X)+(W**2)*Y(X)=0

```
C YB IS THE GUESSED VALUE OF Y AND YBN IS COMPUTED VALUE OFY
      DIMENSION YB(101),SS(101),PI(101),YBN(101),ER(101),
     1X(101)
      REAL KE
    1 CONTINUE
      S=0.
      PI(1)=0.0
      NI=1
      DX=0.01
      BFROG=0.
      DO 15 I=1,101
      X(I)=.01*(I-1)
   15 CONTINUE
      DO 2 I=1,101
      SS(I)=0
      YB(I)=4.*X(I)*(1-X(I))
C NORMALIZE YB
      IF (ABS(YB(I)).LE.BFROG) GO TO 2
      BFROG=ABS(YB(I))
    2 CONTINUE
      DO 16 I=1,101
      YB(I)=YB(I)/BFROG
   16 CONTINUE
    3 CONTINUE
      DO 4 I=1,100
      SO =S
      S=S+(YB(I)+YB(I+1))*DX/2.
      SS(I+1)=SS(I)+(SO/(1+X(I))+S/(1+X(I+1)))*DX/2.
      PI(I+1)=(1./(1+X(I))+1./(1+X(I+1)))*DX/2.+PI(I)
    4 CONTINUE
C NOW GET NORM OF THE NEW YB
      BFROG=0.
      DO 5 I=1,101
      YBN(I)=PI(I)*SS(101)/PI(101)-SS(I)
      IF (ABS(YBN(I)).LE.BFROG) GO TO 5
      BFROG=ABS(YBN(I))
    5 CONTINUE
      OMB=SQRT(1./BFROG)
      ERR=0.
      DO 6 I=1,101
      YBN(I)=YBN(I)/BFROG
      ER(I)=YB(I)-YBN(I)
      IF (ABS(ER(I)).LT.ERR) GO TO 6
      ERR=ABS(ER(I))
    6 CONTINUE
      KE=0.
      PE=0.
    7 CONTINUE
C CHECK TO SEE HOW STODOLA COMPARES WITH RAYLEIGH QUOTIENT
      DO 8 I=1,100
      KE=KE+(YBN(I)**2+YBN(I+1)**2)*DX/2
      PE=PE+(1+X(I+1))*((YBN(I+1)-YBN(I))/DX)**2*DX
    8 CONTINUE
      OMB2=SQRT(PE/KE)
      IF (NI.LT.7) GO TO 12
    9 WRITE(6,10) NI,OMB,ERR,PE,KE,OMB2,(X(I),YBN(I),
     1I=1,101,10)
   10 FORMAT ('1'///////,' ',15X,'NI=',I2,', OMEGA =',F12.6,
```

```
  1/15X,'THE MAXIMUM DIFFERENCE BETWEEN YB AND YBN IS'
  2F12.6,/15X,'PE,KE,AND ROOT (PE/KE) ARE',3F12.6,/
  3/20X,'X',10X,'YBN',//,(12X,2F12.6//))
12 CONTINUE
   DO 11 I=1,101
   YB(I)=YBN(I)
11 CONTINUE
   NI=NI+1
   IF (NI.LT.25) GO TO 270
   GO TO 999
270 IF (ERR.GT.(.0001)) GO TO 3
   GO TO 999
999 WRITE (6,998)
998 FORMAT ('1',2X)
   STOP
   END
```

COMPUTER PROGRAM 4.

NEWTON'S METHOD APPLIED TO EQN (1.1)

(1+X)*Y''(X)+Y'(X)+(W**2)*Y(X)=0

```
      DIMENSION X(101),Y(101),YP(101),YO(101),YPO(101)
      DO 1 I=1,101
      X(I)=.01*(I-1)
    1 CONTINUE
      Y(1)=0.
      YP(1)=1.
      YPN=YP(1)
      DX=0.01
      NI=1
      YO(1)=0
      YPO(1)=0
      YPON=YPO(1)
C GET MODE AND OMEGA BY NEWTONS METHOD. FIRST GUESS OM = 2.0
C USE RUNGA KUTTA 2ND ORDER INTEGRATION ROUTINE
      OM=2.0
  100 DO 2 I=1,100
      J=-1
      XN=X(I+1)
      YN=Y(I)
      YON=YO(I)
  200 YPN=YP(I)-(OM**2)*(Y(I)/(1+X(I))+YN/(1+XN))*DX/2
     1-(YP(I)/(1+X(I))+YPN/(1+XN))*DX/2
      YN=Y(I)+(YP(I)+YPN)*DX/2
      YPON=YPO(I)-(YPO(I)/(1+X(I))+YPON/(1+XN))*DX/2-(OM**2)
     1*(YO(I)/(1+X(I))+YON/(1+XN))*DX/2-2*OM*(Y(I)/(1+X(I))
     2+YN/(1+XN))*DX/2
      YON=YO(I)+(YPO(I)+YPON)*DX/2
  300 IF (J.GT.0) GO TO 400
      J=1
      GO TO 200
  400 YP(I+1)=YPN
      Y(I+1)=YN
      YO(I+1)=YON
      YPO(I+1)=YPON
    2 CONTINUE
      IF (NI.LT.3) GO TO 11
      WRITE (6,10) NI,OM,(X(I),Y(I),I=1,101,10)
   10 FORMAT ('1',////////,'0',15X,'THIS IS PATH NO.',I2,
     1'. OMEGA= ',F12.5,//,18X,'X',11X,'Y',//(10X,2F12.6//))
   11 CONTINUE
      NI=NI+1
C  IF ADMISSABLE QUIT
      IF (Y(101).LT.(0.0001)) GO TO 500
C CORRECT OMEGA
      OM=OM-Y(101)/YO(101)
C IF NI.GT.25 QUIT
      IF (NI.LT.25) GO TO 100
  500 CONTINUE
      STOP
      END
```

COMPUTER PROGRAM 5.

RAYLEIGH-RITZ METHOD APPLIED TO

BESSEL'S EQUATION OF ORDER ZERO

```
      DIMENSION A(3,3),B(3,3),AX(6)   ,BX(6)   ,R(3,3),RLB(3),
     1T(3,3),ALPHA(3,3),AP(4,4),TT(3,3),S(3,3),D(3),C(3),
     2BXD(3),RLBD(3),AQ(4,4),BP(4,4),CM(4,3),CMT(3,4),
     3BCM(4,3),ACM(4,3),P(101),PH(4,101),PHP(4,101),FA(101),
     4FB(101),EM(101),Y(101),Z(101)
      REAL LEV
      X=0.
      DO 1 I=1,101
      P(I)=X
      EM(I)=X
C DEFINITION OF ELEMENTS OF MATRIX B
      PH(1,I)=1.
      PH(2,I)=X**2
      PH(3,I)=((X-.8+ABS(X-.8))/2.)**2
      PH(4,I)=((X-1.6+ABS(X-1.6))/2.)**2
C DEFINITION OF ELEMENTS OF MATRIX A
      PHP(1,I)=2.*X
      PHP(2,I)=X-.8+ABS(X-.8)
      PHP(3,I)=X-1.6+ABS(X-1.6)
   92 X=X+0.02405
    1 CONTINUE
      DX=0.02405
C COMPUTE AP(I,K) AND BP(I,K)
      DO 2 I=1,4
       DO 3 K=1,4
      FBINT=0.
      FB(1)=EM(1)*PH(I,1)*PH(K,1)
      DO 4 J=2,101
      FB(J)=EM(J)*PH(I,J)*PH(K,J)
      FBINT=FBINT+(FB(J)+FB(J-1))*DX/2
    4 CONTINUE
      BP(I,K)=FBINT
    3 CONTINUE
    2 CONTINUE
      DO 600 I=1,3
      DO 601 K=1,3
      FAINT=0.
      FA(1)=P(1)*PHP(I,1)*PHP(K,1)
      DO 602 J=2,101
      FA(J)=P(J)*PHP(I,J)*PHP(K,J)
      FAINT=FAINT+(FA(J)+FA(J-1))*DX/2
  602 CONTINUE
      AP(I,K)=FAINT
      A(I,K)=AP(I,K)
  601 CONTINUE
  600 CONTINUE
C INITIALIZE CM
      DO 51 KA=1,4
      DO 52 KB=1,3
      CM(KA,KB)=0.0
   52 CONTINUE
   51 CONTINUE
C EVALUATION OF THE TRANSFORMATION MATRIX CM
      DO 53 KA=1,3
      CM(KA+1,KA)=1.0
   53 CONTINUE
      CM(1,1)=-5.784
      CM(1,2)=-2.576
      CM(1,3)=-0.648
      CALL GMTRA(CM,CMT,4,3)
C MAKE TRANSFORMATION CM(TRANSPOSED)*B*CM
      CALL GMPRD(BP,CM,BCM,4,4,3)
      CALL GMPRD(CMT,BCM,B,3,4,3)
      NC=1
```

```
C INITIALIZE BX FOR THE INPUT TO EIGEN ROUTINE
      DO 201 J=1,3
      DO 202 I=1,3
      BX(NC)=B(I,J)
      NC=NC+1
      IF (I.GE.J) GO TO 201
  202 CONTINUE
  201 CONTINUE
      CALL EIGEN (BX,R,3,0)
      IF(BX(6).LT.0.) GO TO 806
C RLB(I) IS 1/SQRT OF E. VAL. OF B
      KK=1
      DO 8 II=1,3
      BXD(II)=BX(KK)
      KK=KK+II+1
    8 CONTINUE
      DO 10 II=1,3
      DO 9 JJ=1,3
      B (II,JJ)=0.
    9 CONTINUE
   10 CONTINUE
      DO 6 I =1,3
      RLBD(I)=SQRT(BXD(I))
      RLB(I)=1.0/RLBD(I)
      B (I,I)=RLB(I)
    6 CONTINUE
C MULTIPLY R(I,J)*B (I,J)=T(I,J)
      CALL GMPRD(R,B ,T,3,3,3)
C TRANSPOSE T=TT
      CALL GMTRA(T,TT,3,3)
C MULTIPLY A*T=AQ
      CALL GMPRD(A,T,AQ,3,3,3)
C MULTIPLY TT*AQ=ALPHA
      CALL GMPRD(TT,AQ,ALPHA,3,3,3)
C FIND E.VAL. AND E.VEC. OF ALPHA
      NC=1
      DO 210 J=1,3
      DO 211 I=1,3
      AX(NC)=ALPHA(I,J)
      NC=NC+1
      IF(I.GE.J) GO TO 210
  211 CONTINUE
  210 CONTINUE
      CALL EIGEN (AX    ,S,3,0)
C LEV IS THE LOWEST EIGIN VALUE
      LEV=SQRT(AX(6))
      DO 7 I=1,3
      D(I)=S(I,3)
    7 CONTINUE
C MULTIPLY T*D TO GET VALUES OF C
      CALL GMPRD(T,D,C,3,3,1)
      CONE=-5.784*C(1)-2.576*C(2)-0.648*C(3)
      Z(1)=0.0
      DO 64 I=2,101
      Z(I)=0.02405*(I-1)
   64 CONTINUE
      DO 65 I=1,101
      Y(I)=CONE*PH(1,I)+C(1)*PH(2,I)+C(2)*PH(3,I)+C(3)*
     1PH(4,I)
   65 CONTINUE
      WRITE (6,109)LEV,CONE,C(1),C(2),C(3),(Z(I),Y(I),
     1I=1,101,10)
  109 FORMAT ('1'////////,'0',15X,'THE SMALLEST EIGENVALUE',
     1F15.6//,'0',15X,'VALUES OF C1,C2,C3,AND C4'//,'0',15X
     2,'C1= ',F15.6//,'0',15X,'C2=',F15.6//,'0',15X,'C3=',
     3F15.6//,'0',15X,'C4= ',F15.6//,'0',15X,
     4'VALUES OF Y(X) AT ONE-TENTH INTERVAL'//,
     5'0',19X,'X',14X,'Y'//('          ',2F15.6//))
  806 STOP
      END
```

COMPUTER PROGRAM 6.

STODOLA'S METHOD APPLIED TO

BESSEL'S EQUATION OF ORDER ZERO

```
C YB IS THE GUESSED VALUE OF Y AND YBN IS COMPUTED VALUE OFY
      DIMENSION YB(101),SS(101),PI(101),YBN(101),ER(101),
     1X(101)
      REAL KE
    1 CONTINUE
      S=0.
      NI=1
      DX=0.02405
      BFROG=0.
      DO 15 I=1,101
      X(I)=0.02405*(I-1)
   15 CONTINUE
      DO 2 I=1,101
      SS(I)=0
      YB(I)=(1-X(I)/2.405)**2
C NORMALIZE YB
      IF (ABS(YB(I)).LE.BFROG) GO TO 2
      BFROG=ABS(YB(I))
    2 CONTINUE
      DO 16 I=1,101
      YB(I)=YB(I)/BFROG
   16 CONTINUE
    3 CONTINUE
      SS(1)=0.
      SS(2)=YB(2)*DX/2
      S=0.
      S=S+(YB(1)*X(1)+X(2)*YB(2))*DX/2
      DO 4 I=2,100
      SO =S
      S=S+(YB(I)*X(I)+X(I+1)*YB(I+1))*DX/2
      SS(I+1)=SS(I)+(SO/X(I)+S/X(I+1))*DX/2
    4 CONTINUE
C NOW GET NORM OF THE NEW YB
      BFROG=0.
      DO 5 I=1,101
      YBN(I)=SS(101)-SS(I)
      IF (ABS(YBN(I)).LE.BFROG) GO TO 5
      BFROG=ABS(YBN(I))
    5 CONTINUE
      OMB=SQRT(1./BFROG)
      ERR=0.
      DO 6 I=1,101
      YBN(I)=YBN(I)/BFROG
      ER(I)=YB(I)-YBN(I)
      IF (ABS(ER(I)).LT.ERR) GO TO 6
      ERR=ABS(ER(I))
    6 CONTINUE
      KE=0.
      PE=0.
    7 CONTINUE
C CHECK TO SEE HOW STODOLA COMPARES WITH RAYLEIGH QUOTIENT
      DO 8 I=1,100
      KE=KE+(X(I)*YBN(I)**2+X(I+1)*YBN(I+1)**2)*DX/2
      PE=PE+X(I)*((YBN(I+1)-YBN(I))/DX)**2*DX
    8 CONTINUE
      OMB2=SQRT(PE/KE)
      IF (NI.LT.7) GO TO 12
    9 WRITE(6,10) NI,OMB.ERR,PE,KE,OMB2,(X(I),YBN(I),
     1I=1,101,10)
```

```
10  FORMAT ('1'///////,'0',15X,'NI=',I2,'. OMEGA=',F12.6,
   1/15X,'THE MAXIMUM DIFFERENCE BETWEEN YB AND YBN IS'
   2F12.6,/15X,'PE,KE,AND ROOT (PE/KE) ARE',3F12.6,/
   3/20X,'X',10X,'YBN',//,(12X,2F12.6//))
 12 CONTINUE
    DO 11 I=1,101
    YB(I)=YBN(I)
 11 CONTINUE
    NI=NI+1
    IF (NI.LT.25) GO TO 270
    GO TO 999
270 IF (ERR.GT.(.0001)) GO TO 3
    GO TO 999
999 WRITE (6,998)
998 FORMAT ('1',2X)
    STOP
    END
```

COMPUTER PROGRAM 7.

NEWTON'S METHOD APPLIED TO

BESSEL'S EQUATION OF ORDER ZERO

```
      DIMENSION X(101),Y(101),YP(101),YO(101),YPO(101)
     1,YO1(101),YO2(101)
      DO 1 I=1,101
      X(I)=0.02405*(I-1)
    1 CONTINUE
      Y(1)=1.0
      YP(1)=0.0
      YPN=YP(1)
      DX=0.02405
      NI=1
      YO(1)=0.
      YPO(1)=0.
      YPON=YPO(1)
C GET MODE AND OMEGA BY NEWTONS METHOD. FIRST GUESS OM = 2.0
C USE RUNGA KUTTA 2ND ORDER INTEGRATION ROUTINE
      OM=1.4
  100 XN=X(2)
      YN=Y(1)
      YON=YO(1)
      YPN=YP(1)-(OM**2)*(Y(1)+YN)*DX/2.-(-.5*(OM**2)*Y(1)
     1-.5*OM**2)*DX/2.
      I=1
      YN=Y(1)+(YP(1)+YPN)*DX/2.
      YPN=YP(1)-(OM**2)*(Y(1)+YN)*DX/2.-(-.5*(OM**2)*Y(1)
     1+YPN/XN)*DX/2.
      YN=Y(1)+(YP(1)+YPN)*DX/2
      YPON=YPO(1)-(-OM*Y(1)-OM)*DX/2.-(OM**2)*(YO(1)+YON)*DX
     1/2.-2.*OM*(Y(1)+YN)*DX/2.
      YON=YO(1)+(YPO(1)+YPON)*DX/2.
      YPON=YPO(1)-(-OM*Y(1)+YPON/XN)*DX/2.-(OM**2)*(YO(1)
     1+YON)*DX/2.-2.*OM*(Y(1)+YN)*DX/2.
      YON=YO(1)+(YPO(1)+YPON)*DX/2.
      YP(2)=YPN
      Y(2)=YN
      YO(2)=YON
      YPO(2)=YPON
      DO 2 I=2,100
      J=-1
      YN=Y(I)
      YON=YO(I)
  200 YPN=YP(I)-(OM**2)*(Y(I)+YN)*DX/2.-(YP(I)/X(I)+YPN/XN)*
     1DX/2.
      YN=Y(I)+(YP(I)+YPN)*DX/2.
      YPON=YPO(I)-(YPO(I)/X(I)+YPON/XN)*DX/2.-(OM**2)*(YO(I)
     1+YON)*DX/2.-2.*OM*(Y(I)+YN)*DX/2.
      YON=YO(I)+(YPO(I)+YPON)*DX/2.
      IF (I.GT.5) GO TO 54
   54 CONTINUE
  300 IF (J.GT.0) GO TO 400
      XN=X(I+1)
      J=1
      GO TO 200
  400 YP(I+1)=YPN
      Y(I+1)=YN
      YO(I+1)=YON
      YPO(I+1)=YPON
    2 CONTINUE
      IF (NI.LT.4) GO TO 11
      WRITE (6,10) NI,OM,(X(I),Y(I),I=1,101,10)
   10 FORMAT ('1',////////,' ',15X,'THIS IS PATH NO.',I2,
     1'. OMEGA= ',F12.5,//,18X,'X',11X,'Y',//(10X,2F12.6//))
   11 CONTINUE
```

```
      NI=NI+1
C   IF ADMISSABLE QUIT
      IF (ABS(Y(101)).LT.(0.00001)) GO TO 15
C CORRECT OMEGA
      OM=OM-Y(101)/YO(101)
C IF NI.GT.25 QUIT
      IF (NI.LT.25) GO TO 100
   15 CONTINUE
      STOP
      END
```

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3. ABSTRACT

Three techniques for finding the eigenvalues and eigenfunctions are investigated. A typical problem involves a linear homogeneous differential equation with variable coefficients of the form

$$P(x)\, y''(x) + P'(x)\, y'(x) + \omega^2 M(x) = 0 \qquad (1)$$

The functions $P(x)$ and $M(x)$ are functions which are positive, or have at most isolated zeroes on the fundamental interval $(0,L)$; ω is a parameter. Appropriate end conditions are specified so that the problem is self-adjoint.

The three methods are: Newton's method, Stodola's method, and the Rayleigh-Ritz method. The methods are derived and a computer solution by each method is included in the paper. A second problem involving Bessel's equation of order zero is solved using each method and a comparison of the eigenvalues and eigenfunctions is made with tabulated values.

The results indicate that Newton's method is to be preferred usually.

KEY WORDS	LINK A		LINK B		LINK C	
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